Int. J. Solids Structures, 1977, Vol. 13, pp. 845–854. Pergamon Press. Printed in Great Britain

IMPULSIVE LOADING OF IDEAL FIBRE-REINFORCED RIGID-PLASTIC BEAMS—III

CANTILEVER BEAM

LINDA SHAW and A. J. M. SPENCER
Department of Theoretical Mechanics, University of Nottingham, England

(Received 16 November 1976; revised 3 March 1977)

Abstract—The theory outlined in Part I is applied to the problem of a cantilever beam struck transversely at any point by a mass which subsequently adheres to the beam. In the subsequent motion, slope and velocity discontinuities propagate outwards from the point of impact. Solutions for the velocity and deflection of the various segments of the beam are obtained for the case of linear strain-hardening, and simpler approximate solutions are derived for the case of low impact velocity and/or slight strain-hardening. The discontinuity propagating towards the free end of the beam always comes to rest before it reaches this end, but for sufficiently high values of impact mass and velocity, and a strain-hardening parameter, one or more reflections of the discontinuity may occur at the fixed end of the beam and at the point of impact.

1. INTRODUCTION

In this paper we apply the theory of ideal fibre-reinforced rigid-plastic beams to the problem of a cantilever beam struck by a mass at any point. The special case of a cantilever beam struck at its tip was solved in Part II[2] and has also been considered by Jones[3] for the case of a long beam and linear strain-hardening.

For consistency of notation with Parts I[1] and II[2], it is again supposed that the beam initially lies along the X-axis from $X = -L$ to $X = L$. The end $X = -L$ is constrained so as to be fixed in position and unable to rotate, and the end $X = L$ is free. At time $T = 0$ the beam is struck at the point $X = X_0 = x_0L$ by a mass 2M moving with speed V_0 in the Y-direction. The mass subsequently adheres to the beam. We seek solutions in which, for some period after the impact, discontinuities in slope and velocity propagate outwards from the point of impact. At a time T during this interval, the rightward-moving discontinuity is at $X_0 + A(T)$, and the leftward-moving discontinuity is at $X_0 - B(T)$. The segment $-L \leq X < X_0 - B(T)$ is at rest; the segment $X_0 - B(T) < X < X_0 + A(T)$ moves, with the mass, as a rigid body with speed V_1, and the segment $X_0 + A(T) < X < L$ moves as a rigid body with speed V_2. The assumed configuration shortly after the impact is illustrated in Fig. 1. In Fig. 2 we show, for the case of linear strain-hardening, the trajectories of the propagating discontinuities in the (x, t) plane where, as in Parts I and II, $x = X/L$ and $t = TV_0/L$.

2. SOLUTION FOR $\omega t < \beta a_f$

The governing equations are the equations of motion of the segments $X_0 - B(T) < X < X_0 + A(T)$ and $X_0 + A(T) < X < L$, and the kinematic and dynamic jump conditions at $X = X_0 - B(T)$ and $X = X_0 + A(T)$. It is straight-forward to write these equations down for a general strain-hardening relation, or for any special relation such as that given by (2.8) of Part I. However, unlike the cases discussed in Parts I and II, we have been unable to obtain analytical solutions of these equations for non-linear strain-hardening relations (except in the case of the

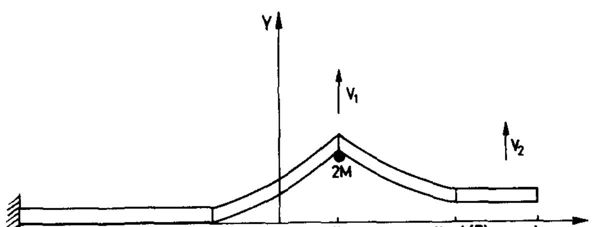

Fig. 1. Impact of a cantilever beam. Assumed form of deformation shortly after impact.

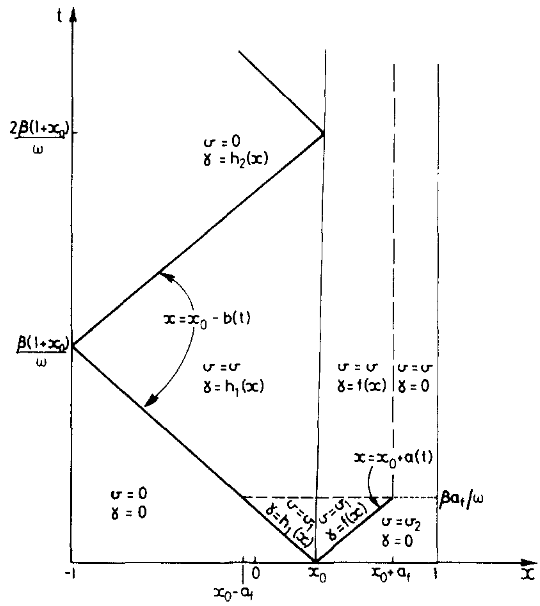

Fig. 2. Impact of a cantilever beam. Trajectories of the discontinuities in the (x, t) plane for linear strain-hardening.

cantilever beam struck at its tip, which is equivalent to the problem discussed in Part II with $X_0 = 0$). We shall therefore consider only linear strain-hardening, so that

$$Q_p = Q_0 + Q_1|\gamma|, \tag{2.1}$$

where we employ the same notation as in Parts I and II. We do, however, note the following relation, which is valid for any strain-hardening rule, and expresses balance of linear momentum of the mass and beam,

$$m(A+B)V_1 + m(L - X_0 - A)V_2 + 2M(V_1 - V_0) = -Q_0T, \tag{2.2}$$

where Q_0 here denotes the initial yield shear force.

We employ the non-dimensional variables defined in Part I (eqns (2.14) and (2.16), with $n = 1$), together with

$$b = B/L, \quad v_1 = V_1/V_0, \quad v_2 = V_2/V_0. \tag{2.3}$$

We also denote the value of γ in the deformed segment, which moves as a rigid body, by

$$\gamma = \begin{cases} f(x) & x_0 < x < x_0 + a(t), \\ h_1(x) & x_0 - b(t) < x < x_0. \end{cases} \tag{2.4}$$

For linear strain-hardening, the discontinuities propagate with constant speed, so that

$$a = b = \omega t/\beta \tag{2.5}$$

in this stage of the deformation.

The equations of motion of $x_0 - b(t) < x < x_0 + a(t)$ and of $x_0 + a(t) < x < 1$ are

$$\beta^2(a + b + 2\alpha)\dot{v}_1 = -2 - \omega^2\{-f(x_0 + a) + h_1(x_0 - b)\}, \tag{2.6}$$

$$\beta^2(1 - x_0 - a)\dot{v}_2 = 1. \tag{2.7}$$

When a is given by (2.5), the kinematic and dynamic jump conditions at $x = x_0 + a(t)$ both reduce to

$$\beta^2\dot{a}(v_1 - v_2) = -\omega^2 f(x_0 + a) \tag{2.8}$$

and similarly the jump conditions at $x = x_0 - b(t)$ both become

$$\beta^2\dot{b}v_1 = \omega^2 h_1(x_0 - b). \tag{2.9}$$

The initial conditions are

$$a = 0, \quad b = 0, \quad v_1 = 1, \quad v_2 = 0 \quad \text{at} \quad t = 0. \tag{2.10}$$

Adding (2.6)–(2.9), integrating and introducing the initial conditions gives

$$\omega\beta\{(a + b + 2\alpha)v_1 + (1 - x_0 - a)v_2\} = 2\omega\beta\alpha - b, \tag{2.11}$$

where we have used (2.5) to express t in terms of b, because it is convenient to use b (or a) rather than t as the independent variable. Equation (2.11) is equivalent to (2.2).

By introducing (2.5) in (2.7), integrating and using the initial conditions, there follows

$$\omega\beta v_2 = \log\frac{1 - x_0}{1 - x_0 - b}. \tag{2.12}$$

Hence, from (2.5), (2.11) and (2.12)

$$\omega\beta v_1 = \frac{\alpha\omega\beta}{b + \alpha} - \frac{1}{2(b + \alpha)}\left[b + (1 - x_0 - b)\log\left\{\frac{1 - x_0}{1 - x_0 - b}\right\}\right]. \tag{2.13}$$

Then, from (2.8), (2.9), (2.12) and (2.13)

$$\omega^2 f(x) = \frac{-\alpha\omega\beta}{x - x_0 + \alpha} + \frac{1}{2(x - x_0 + \alpha)}\left[x - x_0 + (1 + x - 2x_0 + 2\alpha)\log\left\{\frac{1 - x_0}{1 - x}\right\}\right], \tag{2.14}$$

$$\omega^2 h_1(x) = \frac{\alpha\omega\beta}{x_0 - x + \alpha} - \frac{1}{2(x_0 - x + \alpha)}\left[x_0 - x + (1 + x - 2x_0)\log\left\{\frac{1 - x_0}{1 + x - 2x_0}\right\}\right]. \tag{2.15}$$

Provided that the discontinuity at $x_0 - b$ has not previously reached the fixed end $x = -1$, the discontinuity at $x_0 + a$ vanishes when $v_1 = v_2$. We note that v_1 decreases from its initial value one, and v_2 increases from its initial value zero, and $v_2 \to \infty$ as $a \to 1 - x_0$. Hence, $v_1 = v_2$ for some value a_f of a, such that $a_f < 1 - x_0$; that is, the discontinuity at $x_0 + a$ always vanishes before it reaches the free end $x = 1$. From (2.11) and (2.12), $a = a_f$ is the root of

$$\log\left\{\frac{1 - x_0}{1 - x_0 - a}\right\} = \frac{2\alpha\omega\beta - a}{2\alpha + 1 + a - x_0}. \tag{2.16}$$

Some numerical values of a_f are given in Table 1.

Table 1. Values of a_f for $x_0 = 0.2$

$\alpha\omega\beta$ \ α	0	0.1	0.2	0.3
0.1	0.092	0.082	0.074	0.068
0.2	0.170	0.154	0.140	0.130
0.3	0.236	0.216	0.200	0.184
0.4	0.296	0.272	0.252	0.234

As the two discontinuities travel at the same speed, the discontinuity at $x_0 + a$ vanishes before the one at $x_0 - b$ reaches the fixed end provided that

$$a_f < 1 + x_0. \tag{2.17}$$

Since $a_f < 1 - x_0$, this condition is certainly satisfied if $x_0 > 0$. The case $a_f > 1 + x_0$ appears to have little interest, and henceforth it is assumed that the condition (2.17) is satisfied. The solution given by (2.12) and (2.13) then holds for $0 \leq \omega t \leq \beta a_f$. Correspondingly the expressions (2.14) and (2.15) give the slope of the beam in the ranges $x_0 < x < x_0 + a_f$ and $x_0 - a_f < x < x_0$ respectively.

The deflection $Lu(x, t)$ of the beam is given by

$$u(x, t) = \int_{-1}^{x} \gamma(x, t)\, dx. \tag{2.18}$$

Thus, in this stage of the deformation

$$u = 0, \qquad -1 < x < x_0 - b(t), \tag{2.19}$$

$$u = \int_{x_0 - b}^{x} h_1(x)\, dx, \qquad x_0 - b(t) < x < x_0, \tag{2.20}$$

$$u = \int_{x_0 - b}^{x_0} h_1(x)\, dx + \int_{x_0}^{x} f(x)\, dx, \qquad x_0 < x < x_0 + a(t), \tag{2.21}$$

$$u = \int_{x_0 - b}^{x_0} h_1(x)\, dx + \int_{x_0}^{x_0 + a(t)} f(x)\, dx, \qquad x_0 + a(t) < x < 1. \tag{2.22}$$

The integrals cannot be evaluated as standard functions, although they can be expressed in terms of the dilogarithm function

$$F(z) = -\int_0^z \xi^{-1} \log(1 - \xi)\, d\xi$$

which has been tabulated by Mitchell [4]. Approximate analytical expressions for u in the case $\omega\beta \ll 1$ are given in Section 4.

3. SOLUTION FOR $\beta a_f < \omega t < \beta(1 + x_0)$

Provided that the condition (2.17) is satisfied, the first stage of the deformation terminates at $t = \beta a_f/\omega$. Subsequently the discontinuity at $x_0 - b$ continues to propagate to the left, with the segment $-1 < x < x_0 - b(t)$ at rest, and the segment $x_0 - b(t) < x < 1$ moving as a rigid body with speed vV_0. The configuration is illustrated in Fig. 3, and the trajectories of the discontinuities in the (x, t) plane are shown in Fig. 2. We again denote $\gamma = h_1(x)$ in $x_0 - b < x < x_0$.

Then the governing equations are as follows:

(a) Equation of motion of $x_0 - b(t) < x < 1$:

$$\beta^2(1 - x_0 + b + 2\alpha)\dot{v} = -1 - \omega^2 h_1(x_0 - b); \tag{3.1}$$

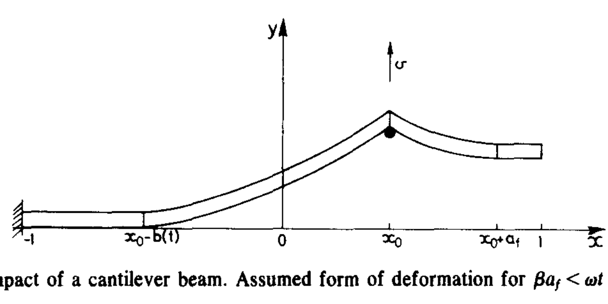

Fig. 3. Impact of a cantilever beam. Assumed form of deformation for $\beta a_f < \omega t < \beta(1 + x_0)$.

(b) Jump condition at $x_0 - b(t)$:

$$\beta^2 \dot{b} v = \omega^2 h_1(x_0 - b); \tag{3.2}$$

together with the relation $b = \omega t/\beta$, and the condition that $v = v_1 = v_2$ at $t = \beta a_f/\omega$, or $b = a_f$.

Adding (3.1) and (3.2) gives

$$\beta^2 \frac{\mathrm{d}}{\mathrm{d}t}\{(1 - x_0 + b + 2\alpha)v\} = -1. \tag{3.3}$$

It remains convenient to use b as the independent variable. By integrating (3.3), inserting the condition at $b = a_f$, and using (2.16), we obtain

$$\omega\beta v = \frac{2\alpha\omega\beta - b}{2\alpha + 1 + b - x_0}. \tag{3.4}$$

Hence, from (3.2), for $x_0 - b < x < x_0 - a_f$

$$\omega^2 h_1(x) = \frac{2\alpha\omega\beta - x_0 + x}{2\alpha + 1 - x}. \tag{3.5}$$

The deflection is still given by (2.19) – (2.22), but now a has the constant value a_f, and $h_1(x)$ takes the form (3.5) in $x_0 - b < x < x_0 - a_f$, and the form (2.15) in $x_0 - a_f < x < x_0$. The expressions for u are again complicated, and approximate expressions are given in Section 4.

This stage of the deformation terminates when either (a) $v = 0$, and the beam comes to rest, or (b) $b = 1 + x_0$, and the discontinuity reaches the fixed end. In case (a), from (3.4), the beam comes to rest when

$$b = 2\alpha\omega\beta. \tag{3.6}$$

Thus the condition for the deformation to be completed before the discontinuity reaches the fixed end is

$$2\alpha\omega\beta < 1 + x_0. \tag{3.7}$$

If (3.7) is satisfied, the deformation terminates when b has the value (3.6) and $t = 2\alpha\beta^2$. Then v is given by (3.4) for $a_f < b < 2\alpha\omega\beta$, and $h_1(x)$ by (3.5) for $x_0 - 2\alpha\omega\beta < x < x_0 - a_f$. The deflection is given by (2.19)–(2.22). As an illustration, the final deflection, when the beam comes to rest, is shown in Fig. 4 for $\alpha = 0.2$ and several values of $\alpha\omega\beta$.

If (3.7) is not satisfied, then (3.4) applies for $a_f < b < 1 + x_0$, and (3.5) for $-1 < x <$

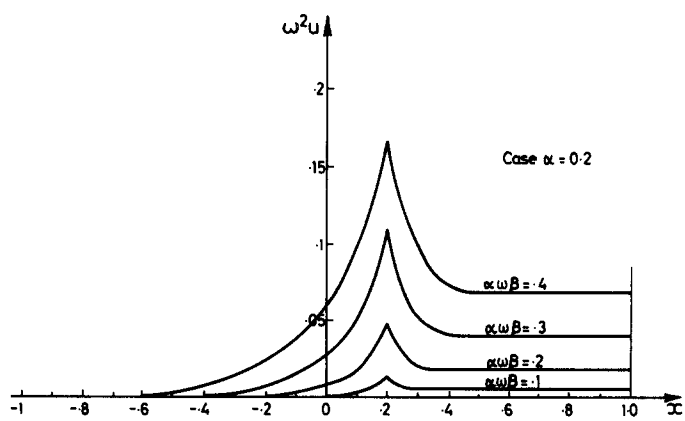

Fig. 4. Final deflection of a cantilever beam for $x_0 = 0.2$, $\alpha = 0.2$.

$x_0 - a_f$. Subsequently, for $\omega t > \beta(1+x_0)$, further deformation takes place, and this is considered in Section 5.

4. SOLUTION FOR $\omega\beta \ll 1$

As in the problems considered in Parts I and II, some simplification arises when $\omega\beta \ll 1$. If $\omega\beta \ll 1$, the condition (3.7) is satisfied except possibly when $1 + x_0 \ll 1$, so that the impact is close to the fixed end, or when $\alpha \gg 1$, which is the case of a heavy striker. We exclude these two cases and assume that (3.7) is satisfied. The deformation is then complete before the discontinuity reaches the fixed end of the beam.

From (2.16), to first order in $\omega\beta$,

$$a_f = \frac{\omega\beta\alpha(1-x_0)}{\alpha+1-x_0}. \tag{4.1}$$

Thus, under the stated assumptions, $a \leq a_f \ll 1$. Hence, from (2.12)–(2.15) and (4.1), to first order in $\omega\beta$,

$$\left.\begin{aligned} \omega\beta v_2 &= \frac{b}{1-x_0}, \\ \omega\beta v_1 &= \frac{\omega\beta\alpha - b}{b+\alpha}, \end{aligned}\right\} \qquad 0 \leq b \leq a_f, \tag{4.2}$$

$$\omega^2 f(x) = \frac{\omega\beta\alpha(x - x_0 - a_f)}{a_f(x - x_0 + \alpha)}, \qquad x_0 < x < x_0 + a_f, \tag{4.3}$$

$$\omega^2 h_1(x) = \frac{\omega\beta\alpha - x_0 + x}{x_0 - x + \alpha}, \qquad x_0 - a_f < x < x_0. \tag{4.4}$$

Hence, from (2.19)–(2.22), the deflection for $\omega t < \beta a_f$ is given by

$$\omega^2 u = \begin{cases} 0, & -1 < x < x_0 - b(t), \\ \alpha(1+\omega\beta)\log\left(\dfrac{b+\alpha}{x_0 - x + \alpha}\right) + x_0 - x - b, & x_0 - b(t) < x < x_0, \\ \alpha(1+\omega\beta)\log\left(\dfrac{b+\alpha}{x - x_0 + \alpha}\right) + x - x_0 - b + \dfrac{\alpha}{1-x_0}\left\{x - x_0 - \alpha\log\left(\dfrac{x - x_0 + \alpha}{\alpha}\right)\right\}, & \\ & x_0 < x < x_0 + a(t), \\ \dfrac{\alpha}{1-x_0}\left\{b - \alpha\log\left(\dfrac{b+\alpha}{\alpha}\right)\right\}, & x_0 + a(t) < x < 1. \end{cases} \tag{4.5}$$

For $\omega t > \beta a_f$, only slight simplification results when $\omega\beta \ll 1$. In this case, from (3.4), to leading order in $\omega\beta$,

$$\omega\beta v = \frac{2\alpha\omega\beta - b}{2\alpha + 1 - x_0}, \qquad a_f \leq b \leq 2\alpha\omega\beta, \tag{4.6}$$

and from (3.5), to first order in $\omega\beta$,

$$\omega^2 h_1(x) = \frac{2\alpha\omega\beta - x_0 + x}{2\alpha + 1 - x_0}, \qquad x_0 - 2\alpha\omega\beta < x < x_0 - a_f. \tag{4.7}$$

Then, from (2.19)–(2.22), with (4.1), (4.3), (4.4) and (4.7), the deflection is given by

$$\omega^2 u = \begin{cases} 0, & -1 < x < x_0 - b(t), \\ \dfrac{(4\alpha\omega\beta - x_0 + x - b)(b - x_0 + x)}{2(2\alpha + 1 - x_0)}, & x_0 - b(t) < x < x_0 - a_f, \\ \dfrac{(4\alpha\omega\beta - a_f - b)(b - a_f)}{2(2\alpha + 1 - x_0)} + \alpha(1 + \omega\beta)\log\left(\dfrac{a_f + \alpha}{x_0 - x + \alpha}\right) + x_0 - x - a_f, & \\ & x_0 - a_f < x < x_0, \\ \dfrac{(4\alpha\omega\beta - a_f - b)(b - a_f)}{2(2\alpha + 1 - x_0)} + \alpha(1 + \omega\beta)\log\left(\dfrac{a_f + \alpha}{x_0 - x + \alpha}\right) + x - x_0 - a_f & \\ \quad + \dfrac{\alpha}{1 - x_0}\left\{x - x_0 - \alpha\log\left(\dfrac{x - x_0 + \alpha}{\alpha}\right)\right\}, & x_0 < x < x_0 + a_f, \\ \dfrac{(4\alpha\omega\beta - a_f - b)(b - a_f)}{2(2\alpha + 1 - x_0)} + \dfrac{\alpha}{1 - x_0}\left\{a_f - \alpha\log\left(\dfrac{a_f + \alpha}{\alpha}\right)\right\}, & x_0 + a_f < x < 1. \end{cases} \tag{4.8}$$

The maximum deflection occurs at $x = x_0$ when $b = 2\alpha\omega\beta$, and is given by

$$\omega^2 u = \frac{(2\alpha\omega\beta - a_f)^2}{2(2\alpha + 1 - x_0)} + \alpha(1 + \omega\beta)\log\left(\frac{a_f + \alpha}{\alpha}\right) - a_f. \tag{4.9}$$

5. SOLUTION AFTER REFLECTION AT $x = -1$

If the condition (3.7) is not satisfied, so that $2\alpha\omega\beta > 1 + x_0$, the discontinuity at $x = x_0 - b$ reaches the fixed end $x = -1$ at time $t = \beta(1 + x_0)/\omega$. In the subsequent motion, the discontinuity at $x = x_0 - b$ is reflected back towards the point of impact, so that at time t its position is given by

$$b = 2(1 + x_0) - (\omega t/\beta) \tag{5.1}$$

for $t > \beta(1 + x_0)/\omega$. The segment $-1 \leq x < x_0 - b$ is at rest, and the segment $x_0 - b < x < 1$, together with the attached mass, moves as a rigid body with speed $V_0 v$. We denote $\gamma = h_2(x)$ in $-1 < x < x_0 - b$. The value of γ in $x_0 - b < x < x_0$ is $h_1(x)$, which is given by (2.15) for $x_0 - a_f < x < x_0$, and by (3.5) for $x_0 - b < x < x_0 - a_f$. The configuration is illustrated in Fig. 5, and the trajectories in the (x, t) plane are shown in Fig. 2.

The governing equations are now:

(a) Equation of motion of $x_0 - b(t) < x < 1$:

$$\beta^2(1 - x_0 + b + 2\alpha)\dot{v} = -1 - \omega^2 h_1(x_0 - b); \tag{5.2}$$

(b) Jump condition at $x_0 - b(t)$:

$$\beta^2 \dot{b} v = \omega^2\{h_1(x_0 - b) - h_2(x_0 - b)\}. \tag{5.3}$$

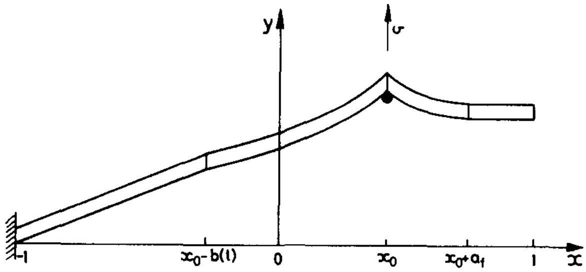

Fig. 5. Impact of a cantilever beam. Form of the deformation after reflection of the discontinuity at the fixed ends.

The initial conditions are, from (3.4),

$$b = 1 + x_0, \quad v = \frac{2\alpha\omega\beta - 1x_0}{2\omega\beta(\alpha - 1)}, \quad \text{when} \quad t = \beta(1 + x_0)/\omega. \tag{5.4}$$

It is again convenient to use b as the independent variable.

By integrating (5.2) subject to the initial conditions (5.4), we obtain

$$\omega\beta v = \frac{2\alpha\omega\beta - 1 - x_0}{2(\alpha + 1)} - \int_b^{1+x_0} \frac{\{1 + \omega^2 h_1(x_0 - b)\}}{1 - x_0 + b + 2\alpha} \mathrm{d}b. \tag{5.5}$$

For $1 + x_0 \geqslant b \geqslant a_f$, (3.5) and (5.5) give

$$\omega\beta v = \frac{\alpha(2\omega\beta + 1) - x_0}{\alpha + 1} - \frac{2\alpha(1 + \omega\beta) + 1 - x_0}{2\alpha + 1 - x_0 + b}. \tag{5.6}$$

This determines v until either the beam comes to rest or b decreases to the value a_f. If the beam is still in motion when $b = a_f$, then subsequently v is still given by (5.5) but eqn (2.15) must be used for $h_1(x_0 - b)$ when b lies between a_f and zero. The resulting expression is complicated and is not stated explicitly.

The slope $h_2(x)$ of the beam behind the first reflected discontinuity is given by (5.3). For $-1 < x < x_0 - a_f$, (3.5), (5.3) and (5.6) give

$$\omega^2 h_2(x) = \frac{2\alpha\omega\beta - (1 + x_0)}{\alpha + 1}, \tag{5.7}$$

so that the slope of the beam is constant behind the reflected discontinuity. We note that this result is consistent with (4.14) of Part II in the case of impact at the tip of the beam, for which $x_0 = 1$. Since $h_2(x)$ is constant, the deflection for $-1 < x < x_0 - b < x_0 - a_f$ is easily found from (5.7). For $x > x_0 - b$ and $b > a_f$, the deflection differs from that given by (4.8) only by a rigid body displacement such that u is continuous at $x = x_0 - b$.

For sufficiently large values of α, ω and β, further reflections of the discontinuity will occur alternately at $x = x_0$ and $x = -1$ until the beam eventually comes to rest.

6. DISCUSSION

The theoretical model employed in this series of papers is a very idealised one, and it is appropriate to discuss the results in relation to the assumptions made. These assumptions fall into four categories.

(a) *Rigid-plastic, time independent behaviour*

This assumption has frequently been made in studies of the dynamic behaviour of isotropic beams. Although elastic and strain-rate effects are often significant, it has been found that the rigid-plastic theory often predicts reasonable results provided that the strains are moderately large. It is reasonable to hope that rigid-plastic theory may have a similar degree of validity in the problems considered here.

(b) *Ideal fibre-reinforced material*

The assumption that the beam is inextensible in its axial direction implies that the deflection arises entirely through shear deformation and that the familiar flexural deformation mode cannot occur. For elastic–plastic materials with small but finite extensibility characterised by a Young's Modulus E and shear yield stress k, the condition for shear deformation to predominate over flexural deformation is, for a rectangular cantilever beam of length $2L$ and depth H, loaded at its tip,

$$\left(\frac{H}{2L}\right)^2 \gg \frac{k}{E|\gamma_0|}, \tag{6.1}$$

where γ_0 is the slope predicted by the inextensible theory. We envisage applications to materials for which the yield stress k is very small compared to E. For validity of the theory it is clearly desirable for γ_0 to be moderately large. For given k, E and γ_0, (6.1) estimates the range of depth to span ratios for which the theory may be expected to apply.

(c) *Small strain and small deflection*

In formulating the theory it has been assumed that the strains are small and geometry changes can be neglected. These assumptions to some extent conflict with (a) and (b) above; the conflict with (a) is present also in the theory for isotropic materials. In the solutions given in these papers, the maximum value Lu_f of the deflection is such that

$$u_f \simeq \frac{1}{2}\alpha\beta^2 = \frac{1}{2}\frac{MV_0^2}{Q_0L}. \tag{6.2}$$

Thus, for the deflection to be small everywhere, it is necessary that the kinetic energy lost in the impact is small compared to Q_0L.

In the solutions given here, the maximum value of γ occurs at the point of impact and is of order

$$\frac{\beta^2}{\omega\beta^q} = \left\{\frac{mV_0^2}{Q_1}\right\}^{1/(n+1)}. \tag{6.3}$$

Thus, if the strain is small everywhere, it is necessary that $mV_0^2 \ll Q_1$. However, even if this condition is violated, the solutions may be valid except in a region in the neighbourhood of the point of impact. We note also that the maximum strain is not sensitive to the mass of the striker, but is strongly dependent on its velocity.

A further restriction on the essentially one-dimensional theory presented here is that the width La of the deforming segment should be large compared to the thickness of the beam. Thus we require

$$a_f \gg \frac{H}{L}.$$

(d) *Mathematical approximations*

For all the problems considered, solutions have been obtained which are exact within the framework of the theory used. However, certain approximations have been used in order to express these solutions in simpler forms. These approximations are not essential, but, when they are valid, it is convenient to use them. Thus, in Part I, we considered the cases of a heavy striker ($\alpha \gg 1$) and a light striker ($\alpha \ll 1$). The validity of these approximations is easily assessed in particular cases. The most useful approximation which has been used is

$$\omega\beta \ll 1, \quad \text{or} \quad \left(\frac{mV_0^2}{Q_0}\right)^{1/2}\left(\frac{Q_1}{Q_0}\right)^{1/2n} \ll 1. \tag{6.4}$$

The advantage of this approximation is that it enables explicit formulae for the deflection to be obtained as, for example, in (4.5) and (4.8) of this paper. However, when (6.4) is not satisfied, the deflection may still be found by straightforward numerical integrations. The validity of (6.4) is also easily assessed in particular cases. The condition (6.4) rules out certain possible deformations; for example, in the case of a supported beam it was shown in Section 3 of Part II that $\omega\beta \ll 1$ implies either $a_1 \ll 1$ or $\alpha \gg 1$. However, the case $\alpha \gg 1$ may, by (6.2), imply that the deflections are unacceptably large. A similar point was made by a rather different argument by Jones ([3], p. 322).

Because of the many idealisations made in formulating the theory, close quantitative agreement between it and experiment is hardly to be expected. It would be of great interest to make comparisons with experiments, but we know of no relevant data. At this stage it may be more profitable to seek to draw conclusions of a more qualitative nature. Two such conclusions are the expressions $\frac{1}{2}L\beta^2 v_f$ ((7.7) of Part I) and $\frac{1}{2}L\alpha\beta^2$ (Section 3 of Part II) for the maximum deflection of the free beam and supported beam (or cantilever struck at its tip) respectively. These results (which are independent of the parameter n) have simple physical interpretations. If, as will frequently be the case, $Q_p - Q_0 \ll Q_0$, then the plastic work done in the deformation is, approximately,

$$W_p = \int_{-L}^{X_0} Q_0 \gamma \, \mathrm{d}X - \int_{X_0}^{L} Q_0 \gamma \, \mathrm{d}X = 2Q_0 L u_f, \tag{6.5}$$

where Lu_f is the final deflection at X_0. Equating W_p to the kinetic energy lost in impact leads immediately to the above expressions for u_f. Since $Q_p \geq Q_0$, (6.5) underestimates W_p, and so this elementary argument gives upper bounds to u_f.

Another general conclusion is that, for fixed V_0, the larger the value of Q_1 the smaller will be the strains and the greater the region of the beam over which the deformation is distributed. Thus, a high rate of strain-hardening (large Q_1) has the effect of absorbing the kinetic energy of the striker as plastic work distributed along a large segment of the beam. For small Q_1, the plastic work is concentrated into a small portion of the beam. In the extreme case of a perfectly plastic solid, which arises in either of the limits $Q_1 \to 0$ or $n \to 0$, the discontinuities do not propagate at all. Some remarks on the case of a perfectly plastic solid were made in reference[1] of Part I.

Acknowledgement—One of the authors (L.S.) thanks the Science Research Council for the award of a Research Studentship.

REFERENCES

1. Linda Shaw and A. J. M. Spencer, Impulsive loading of ideal fibre-reinforced rigid-plastic beams—I. Free beam under central impact. *Int. J. Solids Structures* **13**, 823–831 (1977).
2. Linda Shaw and A. J. M. Spencer, Impulsive loading of ideal fibre-reinforced rigid-plastic beams—II. Beam with end supports. *Int. J. Solids Structure* **13**, 833–844 (1977).
3. N. Jones, Dynamic behaviour of ideal fibre-reinforced rigid-plastic beams, *J. Appl. Mech.* **98**, 319 (1976).
4. K. Mitchell, Tables of the function $\int_0^z (-\log|1-y|/y)\,\mathrm{d}y$, with an account of some properties of this and related functions. *Phil. Mag.* **40**, 351 (1949).